An Application of the Perfectly Matched Layer (PML) Concept to the Finite Element Method Frequency Domain Analysis of Scattering Problems

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***Abstract*—The Perfectly Matched Layer (PML) concept, introduced by Berenger with the aim of synthesizing an absorbing boundary condition (ABC) for the Finite Difference Time Domain (FDTD) method, was recently modified and extended to Finite Element Frequency Domain (FEFD) applications. The modified equations, which neither require the splitting of the field components of interest nor involve negative conductivity parameters, are employed, in this paper, in conjunction with analytic ABC's to obtain a boundary condition (BC) for scattering problems. This BC exhibits an improved performance over a PML medium terminated by a perfect conductor, or an ABC termination alone.**

I. Introduction

In a recent paper, Pekel and Mittra [1] have modified the original PML equations of Berenger [2] to investigate the solution of radiation problems with the finite element frequency domain (FEFD) method. In this paper, we consider the extension of this approach to scattering problems, which requires a considerable and not-so-straightforward modification of the procedure for radiation problems. We show that the number of layers in the PML medium can be reduced to a moderate level, say four, by terminating them with an analytic absorbing boundary condition (ABC).

II. Derivation

We begin with the *transversely-scaled* version of the PML equations [1], [3], which is suitable for implementation in the FEFD formulation and possesses an isotropic right-hand-side

$$\nabla' \times \bar{H} = j\omega\varepsilon_0\left(1 - j\frac{\sigma_2}{\omega\varepsilon_0}\right)\bar{E},$$
$$\nabla' \times \bar{E} = -j\omega\mu_0\left(1 - j\frac{\sigma_2^*}{\omega\mu_0}\right)\bar{H}. \tag{1}$$

The above *curl-like* operator is based on the modified differential operator ∇' defined by

$$\nabla' = \hat{x}\frac{\partial}{\partial x'} + \hat{y}\frac{\partial}{\partial y'} + \hat{z}\frac{\partial}{\partial z} \tag{2}$$

where z is the assumed normal direction of propagation, and

$$\frac{\partial}{\partial x'} = \left(1 - j\frac{\sigma_2}{\omega\varepsilon_0}\right)\frac{\partial}{\partial x} = \left(1 - j\frac{\sigma_2^*}{\omega\mu_0}\right)\frac{\partial}{\partial x},$$
$$\frac{\partial}{\partial y'} = \left(1 - j\frac{\sigma_2}{\omega\varepsilon_0}\right)\frac{\partial}{\partial y} = \left(1 - j\frac{\sigma_2^*}{\omega\mu_0}\right)\frac{\partial}{\partial y}. \tag{3}$$

The parameters σ_2 and σ_2^* represent electric and magnetic conductivities, respectively, associated with the direction of propagation in the original anisotropic version of the PML equations [2], and satisfy the *PML impedance matching condition*, viz., $\sigma_2/\varepsilon_0 = \sigma_2^*/\mu_0$.

The above equations are valid for ordinary elements which lie along the free-space to PML region interface. For elements which lie at the edges (and not at the corners) of a three-dimensional problem domain, where one is effectively faced with two directions of propagation, say x and z, the *curl-like* operator must be based on the following differential operator

$$\nabla' = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y'} + \hat{z}\frac{\partial}{\partial z}. \tag{4}$$

For elements which lie at the corners of the problem domain, with three directions of propagation, the *curl-like* operator reduces to the conventional Cartesian curl operator.

Using (1) as the starting point, we can obtain a modified curl-curl of E equation which can be utilized to derive the following weak form for the PML region [1], [3]

$$\int_{V\,(\mathrm{PML})}\left[\left(1 - j\frac{\sigma_2^*}{\omega\mu_0}\right)^{-1}\nabla' \times \bar{E} \cdot \nabla' \times \bar{\phi}_i - k_0^2\left(1 - j\frac{\sigma_2}{\omega\varepsilon_0}\right)\bar{E}\cdot\bar{\phi}_i\right]dv = \oint_{S\,(\mathrm{PML})}(j\omega\mu_0\hat{n}' \times \bar{H})\cdot\bar{\phi}_i\,ds \tag{5}$$

where $\bar{\phi}_i$ denotes the vectorial weighting functions, and the modified outward normal unit vector $\hat{n}'$ is defined in the same manner as the modified differential operator ∇'. In contrast to the longitudinal (z) scaling case [4], the contribution of the surface integral on the r.h.s. of (5) is zero, since the *scaling* was performed on the transversal components. For a known incident field, the FEFD formulation for the scattered electric field inside the PML region can be written as

$$\int_{V\,(\mathrm{PML})}\left[\left(1 - j\frac{\sigma_2^*}{\omega\mu_0}\right)^{-1}\nabla' \times \bar{E}^{\mathrm{scat}} \cdot \nabla' \times \bar{\phi}_i - k_0^2\left(1 - j\frac{\sigma_2}{\omega\varepsilon_0}\right)\bar{E}^{\mathrm{scat}}\cdot\bar{\phi}_i\right]dv$$
$$= \int_{V\,(\mathrm{PML})}\left[k_0^2\left(1 - j\frac{\sigma_2}{\omega\varepsilon_0}\right)\bar{E}^{\mathrm{inc}}\cdot\bar{\phi}_i - \nabla' \times \left(\left(1 - j\frac{\sigma_2^*}{\omega\mu_0}\right)^{-1}\nabla' \times \bar{E}^{\mathrm{inc}}\right)\cdot\bar{\phi}_i\right]dv. \tag{6}$$

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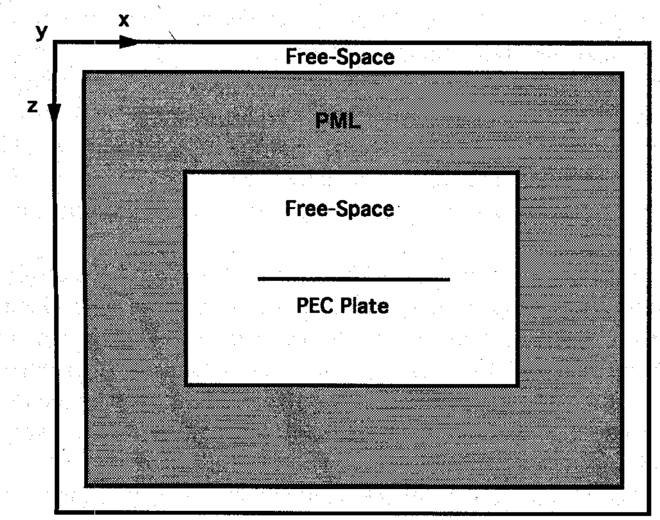


Fig. 1. PEC plate in free-space region surrounded by PML medium and a single layer of free-space terminated by analytic ABC.

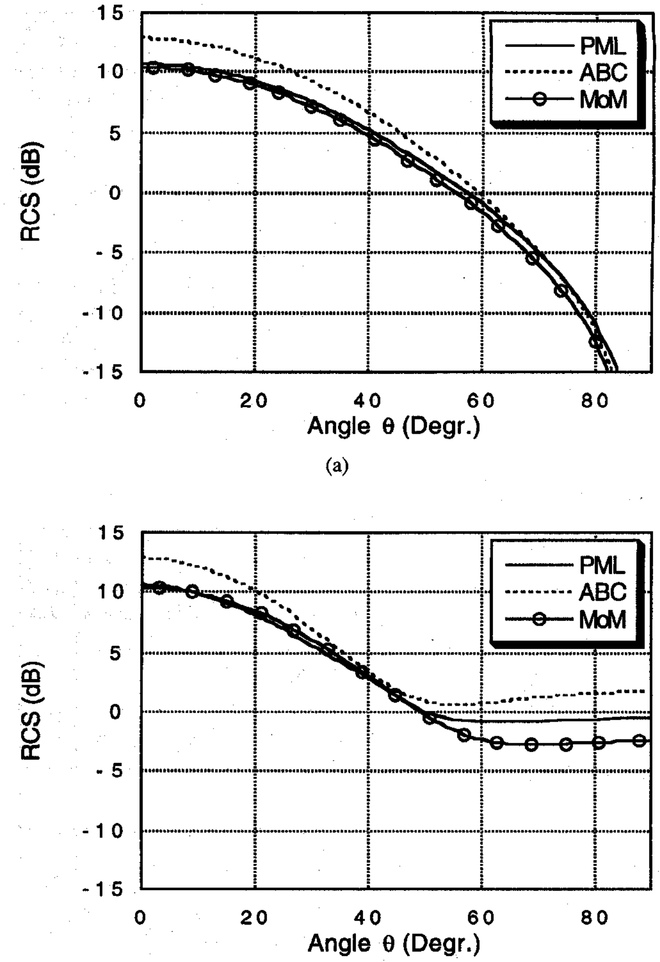
In the non-PML regions, the FEFD formulation reads

$$\int_V \left[\frac{1}{\mu_r}\nabla\times\bar{E}^{\text{scat}}\cdot\nabla\times\bar{\phi}_i - k_0^2\varepsilon_r\bar{E}^{\text{scat}}\cdot\bar{\phi}_i\right]dv + \oint_S\left(\hat{n}\times\frac{1}{\mu_r}\nabla\times\bar{E}^{\text{scat}}\right)\cdot\bar{\phi}_i\,ds = \int_V\left[k_0^2(\varepsilon_r-1)\bar{E}^{\text{inc}}\cdot\bar{\phi}_i - j\omega\mu_0\left(1-\frac{1}{\mu_r}\right)\bar{H}^{\text{inc}}\cdot\bar{\phi}_i\right]dv. \quad (7)$$

III. Numerical Results

The *transversely-scaled* PML equations have been employed, in conjunction with analytic ABC's serving as termination, to investigate the problem of scattering from a $1\lambda_0\times 1\lambda_0$ conducting square plate located in free-space (see Fig. 1), with $\lambda_0 = 1.0$ m. The FEFD approach utilizes a nonuniform, but orthogonal mesh structure comprised of hexahedral, first-order edge elements. The plate is located at the center of the computational domain, and the total thickness of the surrounding PML medium is identical in all three directions. The layers of the PML medium are enclosed by a single layer of free-space terminated by an analytic Engquist-Majda ABC. The size of the interior region is given by $1.3\ \lambda_0\times 1.3\ \lambda_0\times 0.3\lambda_0$ in the x-, y-, and z-directions, respectively, where only 3 layers of elements are utilized between the plate and the PML medium.

For the first numerical example, the PML region has a total thickness (away from the interior region) of $0.14\lambda_0$ comprising four layers, and the total number of unknowns is 69 984. A plane wave is normally incident on the plate, with a y-polarized incident electric field (pointing into the page). Fig. 2(a) shows the bistatic RCS pattern in the principal plane ($\phi = \pi/2$) for a PML medium with a uniform conductivity (σ_2) profile (where $\sigma_2 = 0.015$ S/m) and an analytic ABC termination. The uniform profile has been found [3] to yield more accurate results than the linear or quadratic ones when ABC terminations are used. The calculated RCS pattern is compared to the one obtained with the same ABC termination applied in the absence of the PML medium, and to an MoM reference pattern. A comparison of the bistatic RCS patterns on the orthogonal plane ($\phi = 0$) is given in Fig. 2(b). The presence of the surrounding PML medium gives rise to a clear improvement in the accuracy of the RCS patterns in both planes. Thus, a four-layer PML medium with a uniform conductivity profile and an analytic ABC termination is capable of simulating a reasonably reflectionless and absorptive medium, and leads to an improved boundary condition for the FEFD analysis of scattering problems, requiring an overall smaller problem domain than an ordinary ABC termination applied considerably farther away from the plate to attain the same numerical accuracy.



(b)

Fig. 2. Comparison of calculated RCS patterns (four-layer PML medium with uniform conductivity profile and ABC termination, ABC alone, and MoM) on the (a) principal plane, (b) orthogonal plane.

Next, the four layers of the PML-type medium are used without the ABC termination on the single surrounding layer of free-space while retaining the size of the problem domain. An approximate boundary condition (BC) which assumes a complete absorption the scattered field in the PML region is indirectly enforced as termination by neglecting the surface integral term in (7). Fig. 3(a) and (b) shows the bistatic RCS patterns in the principal and orthogonal planes, respectively,

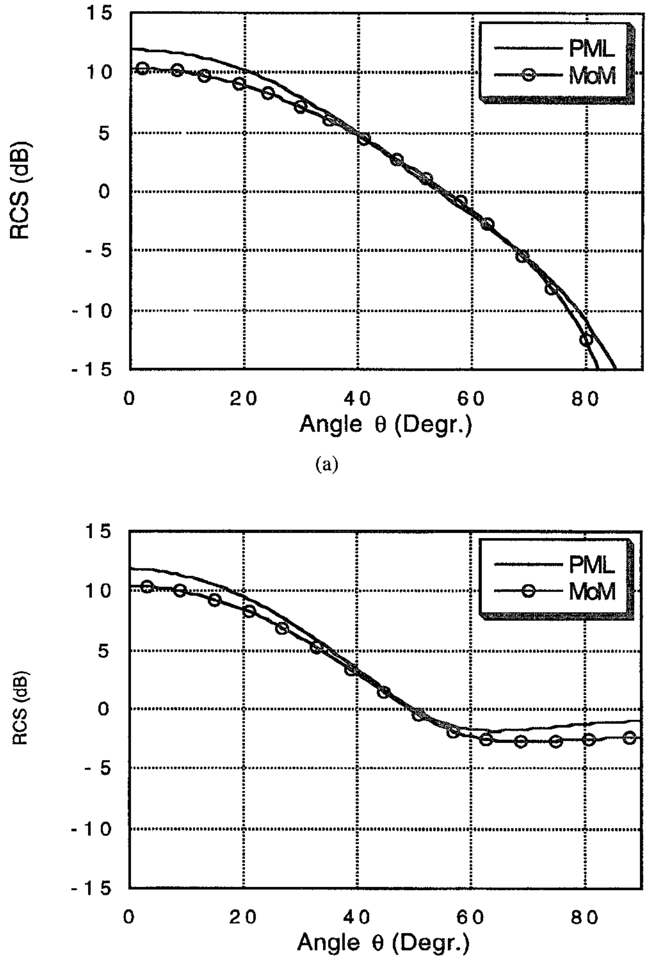


Fig. 3. Comparison of calculated RCS patterns (four-layer PML medium with uniform conductivity profile and approximate BC termination, and MoM) on the (a) principle plane, (b) orthogonal plane.

computed with the FEFD approach utilizing this particular BC, and the deterioration of the patterns when compared to the PML+ABC results is apparent. This leads us to conclude that for the practical case where only a moderate number of layers is employed in the PML medium, the medium should be terminated neither by an approximate boundary condition, nor with a PEC surface, but with an ABC termination instead.

The accuracy of the PML termination can be steadily improved by increasing the number of layers from four, albeit at the expense of increasing the number of unknowns which grows rapidly. However, little or no improvement is achieved by increasing the separation distance of the inner boundary of the PML medium from the plate beyond the three layers employed in this work.

The optimal choice of the conductivity σ_2 often requires a considerable amount of numerical experimentation. Choosing too low a σ_2 results in an insufficient attenuation; on the other hand, a large σ_2 induces a rapid decay of the field in the PML layer and requires a very fine and costly discretization within the PML medium in order to accurately model the field behavior.

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